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Quadratic and Higher-Order Feedback Gains for Control of Nonlinear Systems

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The problem of controlling a nonlinear system optimally in the presence of deterministic disturbances is treated. In particular, optimal feedback control in the vicinity of an optimal nominal trajectory is sought. The control law is to preserve the optimality of the nominal path and provide terminal control to the nominal end point defined by hard constraints on functions of the terminal state. A method of obtaining approximations of any order to such a nonlinear optimal feedback control law is presented. Linear differential equations for the linear and quadratic feedback gains are explicitly given. The equations for the linear gains are homogeneous, whereas those for higher-order gains are nonhomogeneous. The forcing terms are functions of the gains of the next lower order. Closed-form expressions for the linear and quadratic gains for a simple intercept problem and their simulations are presented. The addition of quadratic gains to the linear control approximation generally results in improved control.

I. Introduction and Summary

MOST methods developed to date for the solution of nonlinear optimal feedback control problems are based on linearization about some reference trajectory of the basic nonlinear system. As such, they provide a linear feedback control law valid for small deviations from the reference trajectory. Such methods include the work of Kalman,[1] and Bryson and Denham.[2] Although Kalman considered the linear control problem, his results are applicable to the linearized motion about a reference trajectory of a nonlinear system. In the works just mentioned, the reference trajectory is arbitrary and need not be optimal in any sense.

A different approach was taken by Kelley,[3] Breakwell, Speyer, and Bryson,[4] Dreyfus[5] and this author.[6] These investigators obtained the linear feedback gains for what may be termed neighboring extremal control. Here the reference trajectory of the basic nonlinear system is optimal, and the linear control law is optimal in the same sense to a first approximation.

Recently, Silber[7] developed a control scheme, in which the Lagrange multipliers are considered as control parameters, and obtained differential equations for the coefficients in the Taylor series expansion of the Lagrange multipliers as functions of the state. This permits a high-order approximation to neighboring extremals. Kushner[8] treats the stochastic optimal feedback control problem and presents a method of determining high-order corrections to the optimal deterministic control in the presence of stochastic disturbances.

In the present work we assume that the state of the nonlinear system can be measured exactly. Deviations in the state from an optimal nominal state are taken as the feedback information. We develop a high-order approximation to the nonlinear optimal feedback control law in the vicinity of an optimal nominal trajectory, which is a solution of the Mayer problem in variational calculus. The control scheme is a neighboring extremal control scheme in that it provides terminal control and, in the presence of disturbances, enables the system to follow trajectories that are approximately optimal in the same sense as the nominal trajectory to any order. Differential equations for the linear and quadratic optimal feedback gains are explicitly given. Computational aspects associated with these equations will be treated in a subsequent paper. The linear feedback gains are identical to those developed in Refs. 3–6.

Closed-form expressions for the linear and quadratic optimal feedback control gains for a simple intercept problem are

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given. A number of control simulations are presented and the control effectiveness of the linear and linear plus quadratic feedback gains is compared. In most cases a substantial improvement in control effectiveness is realized with the addition of the quadratic feedback gains.

II. Nonlinear Optimal Feedback Control Problem

In this section we pose a general nonlinear optimal feedback control problem and provide a setting for the analysis that follows. Assume that the dynamical system to be controlled is governed by the differential equation

$$dx/dt = \dot{x} = f(x, u, t) \tag{1}$$

where x is an n vector called the *state* of the system; $u(t)$ is an m vector of *control* variables; f is an n vector; and t is the independent variable time. The vector function f is to possess derivatives of any required† order in all arguments. Assume further that exact physical measurements of the state are available at every instant of time in the interval of interest $[t_0, t_1]$.

It is required to determine a *control law*

$$u = u(x, t) \tag{2}$$

defined on $[t_0, t_1]$ so that the resultant motion of the system (1) initiating at the phase (x°, t_0) which satisfies, for given t_0,

$$\xi[x(t_0), t_0] = 0 \tag{3}$$

passes through

$$\eta[x(t_1), t_1] = 0 \tag{4}$$

and provides a minimum of the scalar function

$$J[x(t_1), t_1] \tag{5}$$

Here ξ is an r vector and η is a q vector $(q < 2n - r + 1)$. We consider only those control problems in which $u(x, t)$ has derivatives of any required order in all arguments.

The solution $u(x, t)$ of the problem just stated is clearly the *optimal feedback control law* since it permits closed-loop operation in the presence of disturbances. At any phase (x, t) the control action to be taken is precisely determined by Eq. (2).

Even with these strong assumptions, the optimal feedback control problem cannot be solved in general when the system equations (1) are nonlinear. In view of this, consider the approximate representation of u in terms of the Taylor series

$$u_i(x, t) = u_i(\bar{x}, t) + \alpha_{ij}(\bar{x}, t)\delta x_j + \tfrac{1}{2}\beta_{jk}{}^i(\bar{x}, t)\delta x_j \delta x_k + \ldots \qquad i = 1, \ldots, m \tag{6}‡$$

where

$$\delta x_i(t) = x_i(t) - \bar{x}_i(t) \tag{7}$$

The expansion is carried out at each point in time with respect to a nominal trajectory $\bar{x}(t)$ initiating at (x°, t_0) with control $u(\bar{x}, t)$. The coefficients appearing in Eq. (6) are the appropriate partial derivatives of the control variables with respect to the state variables and are functions of time only, being evaluated along the nominal trajectory. $u(\bar{x}, t)$ is likewise a function of time only. It is assumed that u is sufficiently differentiable so that it possesses a convergent Taylor series expansion of the order considered.

† Exact differentiability requirements will become apparent later.

‡ Indices repeated in a product imply a summation over the whole range of definition of the index.

Now the problem of determining $u(\bar{x}, t)$ as a function of time, given the system equations (1) and subject to (3) and (4), so as to minimize J, is a Mayer problem in the classical variational calculus, provided it has a solution within the class of controls considered. The control $u(\bar{x}, t)$, the solution of the Mayer problem, is essentially *open loop* since it is only optimal with respect to the initial phase (x°, t_0). The corresponding nominal trajectory $\bar{x}(t)$, around which expansion (6) is carried out, will be called the *optimal nominal trajectory*.

In lieu of determining $u(x, t)$, one might determine $u(\bar{x}, t)$ as a solution of the Mayer problem, and then determine the time-varying coefficients in Eq. (6) so as to satisfy end conditions (4) and provide a minimum of J to a given order of approximation. The coefficients in (6) so determined are the *feedback gains*, which permit closed-loop operation in the presence of disturbances $\delta x_i(t)$ in the vicinity of the optimal nominal trajectory. The approximately optimal feedback control law (6) that results provides neighboring extremal control inasmuch as it provides an approximation to neighboring optimal trajectories (extremals) to a terminal phase (x^1, t_1) defined by (4) which also minimizes J. In other words, the control law (6) provides terminal control as well as enabling the system to follow trajectories that are approximately optimal in the same sense as the nominal trajectory. The determination of the coefficients in Eq. (6) is the subject of the present paper.

III. Optimal Nominal Trajectory

The optimal open-loop control $u(\bar{x}, t)$ is taken as the solution of the Mayer problem, as discussed in the preceding section. The optimal nominal trajectory therefore satisfies the equations of state (1) and the Euler-Lagrange equations in the state

$$\dot{\mu} = -[\partial f/\partial x]^T \mu = -[\partial H/\partial x] \tag{8}$$

and in the controls

$$[\partial f/\partial u]^T \mu = [\partial H/\partial u] = 0 \tag{9}$$

The vector μ is an n vector of Lagrange multipliers, and the matrices

$$\left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad \left[\frac{\partial f}{\partial u}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}$$

are functions of time only, being evaluated along the optimal nominal trajectory. Superscript T denotes the transpose. The Hamiltonian notation is used, where

$$H \equiv \mu^T f \tag{10}$$

and

$$\left[\frac{\partial H}{\partial x}\right] = \begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \vdots \\ \frac{\partial H}{\partial x_n} \end{bmatrix} \qquad \left[\frac{\partial H}{\partial u}\right] = \begin{bmatrix} \frac{\partial H}{\partial u_1} \\ \vdots \\ \frac{\partial H}{\partial u_m} \end{bmatrix}$$

Equations (1, 8, and 9) are subject to boundary conditions (3) and (4) and the transversality conditions resulting from the variational formulation. Of interest in the following are the terminal boundary conditions that may be represented as[6]

$$h = \left[\frac{\eta}{\zeta}\right] = \begin{bmatrix} \eta_1[x(t_1), t_1] \\ \cdot \\ \cdot \\ \cdot \\ \eta_q[x(t_1), t_1] \\ \zeta_{q+1}[x(t_1), \mu(t_1), t_1] \\ \cdot \\ \cdot \\ \cdot \\ \zeta_{n+1}[x(t_1), \mu(t_1), t_1] \end{bmatrix} = 0 \tag{11}$$

where ζ are the conditions arising from transversality and in general involve the terminal values of the Lagrange multipliers as well as the state.

We assume that the matrix of partial derivatives $\partial^2 H/\partial u_i \partial u_j$ is nonsingular along the optimal nominal trajectory. This assures that Eq. (9) can be solved for the control and excludes systems linear in the control, singular extremals, and extremals with corners. (The requirement of no corners is convenient but nonessential.) It is of course assumed that $u(\bar{x}, t)$ does provide a minimum of J.

IV. Feedback Gains for Neighboring Extremal Control

We will now sketch the formal procedure for obtaining differential equations and appropriate boundary conditions for the feedback gains for neighboring extremal control as defined in Sec. II. We will then develop the differential equations and boundary conditions for the linear and quadratic feedback gains.

A. General Considerations

Neighboring extremals to an end point defined by (4) satisfy the equations

$$\dot{\bar{x}} + \dot{\delta x} = f(\bar{x} + \delta x, \bar{u} + \delta u, t) \tag{12}$$

$$\dot{\bar{\mu}} + \dot{\delta\mu} = -[\partial f/\partial x(\bar{x} + \delta x, \bar{u} + \delta u, t)]^T(\bar{\mu} + \delta\mu) \tag{13}$$

$$[\partial f/\partial u(\bar{x} + \delta x, \bar{u} + \delta u, t)]^T(\bar{\mu} + \delta\mu) = 0 \tag{14}$$

subject to the terminal boundary conditions

$$h[\bar{x}(t_1) + dx^1, \bar{\mu}(t_1) + d\mu^1, \bar{t}_1 + dt_1] = 0 \tag{15}$$

where the bar denotes optimal nominal values. The differentials dx^1 and $d\mu^1$ are total changes in the terminal values of the vectors x and μ due to a change in terminal time dt_1 as well as instantaneous variations δx and $\delta\mu$.

A trivial generalization is to consider neighboring extremals to an end point on a neighboring surface by specifying a nonzero $d\eta$ and setting $d\zeta = 0$. This extension will not be explicitly treated here.

Consider now what happens if the state of the system (1) is perturbed at t_0 by an amount $\delta x(t_0) = \delta x^\circ$. In order to meet end conditions (4) at some new time $\bar{t}_1 + dt_1$ the initial values (and time histories) of the Lagrange multipliers μ will have to change. Clearly the state time history and terminal time will change as well. We may, in fact, consider the state, the Lagrange multipliers, and terminal time to be functions of the initial state, and write

$$x(t) = g(t; t_0, x^\circ) \tag{16}$$

$$\mu(t) = v(t; t_0, x^\circ) \tag{17}$$

$$t_1 = w(t_0, x^\circ) \tag{18}$$

Assuming that the functions g, v, and w possess convergent Taylor's series expansions of the required order, we have

$$\delta x_i(t) = \gamma_{ij}(t, t_0)\delta x_j^\circ + \tfrac{1}{2}\epsilon_{jk}{}^i(t, t_0)\delta x_j^\circ \delta x_k^\circ + \ldots \quad i = 1, \ldots, n \tag{19}$$

$$\delta\mu_i(t) = \psi_{ij}(t, t_0)\delta x_j^\circ + \tfrac{1}{2}\theta_{jk}{}^i(t, t_0)\delta x_j^\circ \delta x_k^\circ + \ldots \tag{20}$$

$$dt_1 = \tau_i(t_0)\delta x_i^\circ + \tfrac{1}{2}\kappa_{ij}(t_0)\delta x_i^\circ \delta x_j^\circ + \ldots \tag{21}$$

where the coefficients are the appropriate partial derivatives with respect to the initial state variables. (A sufficient condition for the existence of continuous partial derivatives of g, v, and w of a given order with respect to the initia conditions is the existence of continuous partial derivativesl of the same order of the f_i with respect to the state.)

Further, for given δx_i°, we may differentiate Eqs. (19) and (20) with respect to time and obtain

$$\dot{\delta x}_i(t) = \dot{\gamma}_{ij}(t, t_0)\delta x_j^\circ + \tfrac{1}{2}\dot{\epsilon}_{jk}{}^i(t, t_0)\delta x_j^\circ \delta x_k^\circ + \ldots \tag{22}$$

$$\dot{\delta\mu}_i(t) = \dot{\psi}_{ij}(t, t_0)\delta x_j^\circ + \tfrac{1}{2}\dot{\theta}_{jk}{}^i(t, t_0)\delta x_j^\circ \delta x_k^\circ + \ldots \tag{23}$$

We note that for each i the matrices of elements $\epsilon_{jk}{}^i$, $\theta_{jk}{}^i$, and κ_{jk} are symmetric.

We may now proceed formally as follows. Expand the right-hand sides of Eqs. (12–14) in Taylor's series. Eliminate the variations δu from the results via Eq. (6). Substitute for δx, $\delta\mu$ and their time derivatives from Eqs. (19, 20, 22, and 23). Equate the coefficients of the *independent* perturbations δx_i° in the resultant equations to obtain differential equations for the coefficients in Eqs. (19) and (20), and relations between these coefficients and the feedback gains [coefficients in Eq. (6)]. The boundary conditions for the differential equations thus derived are obtained from Eq. (19), by evaluating it at t_0, and from similar expansion and substitution procedures in Eq. (15).

The procedure outlined in the foregoing is carried out to the second order in Appendix A. The results are discussed in the next section.

B. Differential Equations and Boundary Conditions for Linear and Quadratic Feedback Gains

The differential equations for the linear coefficients in Eqs. (19) and (20), derived in Appendix A, are

$$\begin{aligned} \dot{\gamma} &= \left(\left[\frac{\partial f}{\partial x}\right] - \left[\frac{\partial f}{\partial u}\right]\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left[\frac{\partial^2 H}{\partial u \partial x}\right]\right)\gamma - \left[\frac{\partial f}{\partial u}\right]\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left[\frac{\partial f}{\partial u}\right]^T \psi \\ \dot{\psi} &= \left(\left[\frac{\partial^2 H}{\partial x \partial u}\right]\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left[\frac{\partial^2 H}{\partial u \partial x}\right] - \left[\frac{\partial^2 H}{\partial x^2}\right]\right)\gamma + \left(-\left[\frac{\partial f}{\partial x}\right]^T + \left[\frac{\partial^2 H}{\partial x \partial u}\right]\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left[\frac{\partial f}{\partial u}\right]^T\right)\psi \end{aligned} \tag{24}$$

with boundary conditions

$$\gamma_{ij}(t_0, t_0) = \delta_{ij} \tag{25}$$

The matrices involved in Eq. (24) are defined in Appendix A. The linear feedback gains are given in terms of the solutions of Eq. (24) by

$$\partial h_i/\partial x_j{}^1[\gamma_{jk}(\bar{t}_1, t_0) + f_j{}^1\tau_k] + \partial h_i/\partial\mu_j{}^1\ [\psi_{jk}(\bar{t}_1, t_0) - (\partial H/\partial x_j{}^1)\ \tau_k] + (\partial h_i/\partial\bar{t}_1)\ \tau_k = 0 \tag{26}$$

$$\alpha = -\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left(\left[\frac{\partial f}{\partial u}\right]^T \psi\gamma^{-1} + \left[\frac{\partial^2 H}{\partial u \partial x}\right]\right) \tag{27}$$

We have assumed that the matrices $[\partial^2 H/\partial u^2]$ and γ are nonsingular. (For the significance of nonsingularity of γ, see Ref. 4.)

We observe that the $2n^2$ differential equations (24) are linear and homogeneous with time-varying coefficients since all the partial derivatives are evaluated along the optimal nominal trajectory. These are precisely the equations derived previously by Refs. 3–6 and the linear feedback gains given by Eq. (27) are the same.

Equations (26) represent $n(n+1)$ equations, since h is an $n+1$ component vector, and provide n^2 terminal boundary conditions on the differential equations (24) as well as determine the n linear coefficients $\tau_k(t_0)$ in the expansion for dt_1, Eq. (21). These n^2 terminal boundary conditions, together with the n^2 initial conditions (25), provide the required $2n^2$ boundary conditions for Eqs. (24). It is seen that a two-point boundary value problem has to be solved to determine the linear feedback gains given in Eq. (27). The problem is a linear one, however, and straightforward to solve.[4,6]

The differential equations for the quadratic coefficients in Eqs. (19) and (20) are derived in Appendix A and appear as Eqs. (A14) and (A15). Using the symmetry properties of $\beta_{jk}{}^i$ and of the partial derivatives, these can be written

$$\dot{\epsilon}_{jk}{}^i = \left(\frac{\partial f_i}{\partial x_l} + \frac{\partial f_i}{\partial u_m}\alpha_{ml}\right)\epsilon_{jk}{}^l + \frac{\partial f_i}{\partial u_l}\beta_{mn}{}^l\gamma_{mj}\gamma_{nk} + \frac{\partial^2 f_i}{\partial x_l\partial x_m}\gamma_{lj}\gamma_{mk} + \frac{\partial^2 f_i}{\partial x_l\partial u_m}\alpha_{mn}(\gamma_{lj}\gamma_{nk} + \gamma_{lk}\gamma_{nj}) + \frac{\partial^2 f_i}{\partial u_l\partial u_m}\alpha_{ln}\alpha_{mp}\gamma_{nj}\gamma_{pk} \quad (28)$$

$$-\dot{\theta}_{jk}{}^i = \left(\frac{\partial^2 H}{\partial x_i\partial x_l} + \frac{\partial^2 H}{\partial x_i\partial u_m}\alpha_{ml}\right)\epsilon_{jk}{}^l + \frac{\partial^2 H}{\partial x_i\partial\mu_l}\theta_{jk}{}^l + \frac{\partial^2 H}{\partial x_i\partial u_l}\beta_{mn}{}^l\gamma_{mj}\gamma_{nk} + \frac{\partial^3 H}{\partial x_i\partial x_l\partial x_m}\gamma_{lj}\gamma_{mk} + \frac{\partial^3 H}{\partial x_i\partial x_l\partial\mu_m}(\gamma_{lj}\psi_{mk} + \gamma_{lk}\psi_{mj}) + \frac{\partial^3 H}{\partial x_i\partial x_l\partial u_m}\alpha_{mn}(\gamma_{lj}\gamma_{nk} + \gamma_{lk}\gamma_{nj}) + \frac{\partial^3 H}{\partial x_i\partial\mu_l\partial u_m}\alpha_{mn}(\psi_{lj}\gamma_{nk} + \psi_{lk}\gamma_{nj}) + \frac{\partial^3 H}{\partial x_i\partial u_l\partial u_m}\alpha_{ln}\alpha_{mp}\gamma_{nj}\gamma_{pk} \quad (29)$$

where the quadratic feedback gains ($\beta_{jk}{}^i$) [Eq. (A16)] are given by

$$\left(\frac{\partial^2 H}{\partial u_i\partial x_l} + \frac{\partial^2 H}{\partial u_i\partial u_m}\alpha_{ml}\right)\epsilon_{jk}{}^l + \frac{\partial^2 H}{\partial u_i\partial\mu_l}\theta_{jk}{}^l + \frac{\partial^2 H}{\partial u_i\partial u_l}\beta_{mn}{}^l\gamma_{mj}\gamma_{nk} + \frac{\partial^3 H}{\partial u_i\partial x_l\partial x_m}\gamma_{lj}\gamma_{mk} + \frac{\partial^3 H}{\partial u_i\partial x_l\partial\mu_m}(\gamma_{lj}\psi_{mk} + \gamma_{lk}\psi_{mj}) + \frac{\partial^3 H}{\partial u_i\partial x_l\partial u_m}\alpha_{mn}(\gamma_{lj}\gamma_{nk} + \gamma_{lk}\gamma_{nj}) + \frac{\partial^3 H}{\partial u_i\partial\mu_l\partial u_m}\alpha_{mn}(\psi_{lj}\gamma_{nk} + \psi_{lk}\gamma_{nj}) + \frac{\partial^3 H}{\partial u_i\partial u_l\partial u_m}\alpha_{ln}\alpha_{mp}\gamma_{nj}\gamma_{pk} = 0 \quad (30)$$

serve to determine the quadratic feedback gains $\beta_{jk}{}^i = \beta_{kj}{}^i$.§ Here m is the dimension of the control vector.

Differential equations (28) and (29) are linear, inhomogeneous with time-varying coefficients. The coefficients and forcing terms are functions of the solutions of Eq. (24) and the linear feedback gains, as well as partial derivatives associated with the optimal nominal trajectory, all known functions of time once the linear feedback gains have been obtained. It is easy to see that this is the case in general, i.e., the differential equations for the kth order gains ($k \geq 2$) will be linear, inhomogeneous with time-varying coefficients. The coefficients and forcing terms will depend on the $(k-1)$th order gains.

A linear two-point boundary value problem results for $\epsilon_{jk}{}^i$ and $\theta_{jk}{}^i$ as well. $n^2(n+1)/2$ initial conditions are given by Eqs. (31). Equations (32), which are $n(n+1)^2/2$ in number, provide the required $n^2(n+1)/2$ terminal boundary conditions and determine the $n(n+1)/2$ quadratic coefficients κ_{ij} in the expansion for dt_1, Eq. (21).

The boundary conditions on the $\epsilon_{jk}{}^i$ and $\theta_{jk}{}^i$ [Eqs. (A19) and (A26)] are

$$\epsilon_{jk}{}^i(t_0, t_0) = 0 \quad (31)$$

and

$$\frac{\partial h_i}{\partial x_j{}^1}\Big\{\epsilon_{kl}{}^j(\bar{t}_1, t_0) + f_j{}^1\kappa_{kl} + \frac{\partial f_j}{\partial x_n{}^1}[\gamma_{nk}(\bar{t}_1, t_0)\tau_l + \gamma_{nl}(\bar{t}_1, t_0)\tau_k] + \frac{\partial f_j}{\partial u_m{}^1}\alpha_{mn}(\bar{x}, \bar{t}_1)[\gamma_{nk}(\bar{t}_1, t_0)\tau_l + \gamma_{nl}(\bar{t}_1, t_0)\tau_k] + \left(\frac{\partial f_j}{\partial t} + \frac{\partial f_j}{\partial x_m}f_m + \frac{\partial f_j}{\partial u_m}\dot{u}_m\right)_{\bar{t}_1}\tau_k\tau_l\Big\} + \frac{\partial h_i}{\partial\mu_j{}^1}\Big\{\theta_{kl}{}^j(\bar{t}_1, t_0) - \frac{\partial H}{\partial x_j{}^1}\kappa_{kl} - \frac{\partial^2 H}{\partial x_j{}^1\partial x_m{}^1}[\gamma_{mk}(\bar{t}_1, t_0)\tau_l + \gamma_{ml}(\bar{t}_1, t_0)\tau_k] - \frac{\partial^2 H}{\partial x_j{}^1\partial\mu_m{}^1}[\psi_{mk}(\bar{t}_1, t_0)\tau_l + \psi_{ml}(\bar{t}_1, t_0)\tau_k] - \frac{\partial^2 H}{\partial x_j{}^1\partial u_m{}^1}\alpha_{mn}(\bar{x}, \bar{t}_1)[\gamma_{nk}(\bar{t}_1, t_0)\tau_l + \gamma_{nl}(\bar{t}_1, t_0)\tau_k] - \left(\frac{\partial^2 H}{\partial t\partial x_j} + \frac{\partial^2 H}{\partial x_m\partial x_j}f_m - \frac{\partial^2 H}{\partial\mu_m\partial x_j}\frac{\partial H}{\partial x_m} + \frac{\partial^2 H}{\partial u_m\partial x_j}\dot{u}_m\right)_{\bar{t}_1}\tau_k\tau_l\Big\} + \frac{\partial h_i}{\partial\bar{t}_1}\kappa_{kl} + \frac{1}{2}\frac{\partial^2 h_i}{\partial x_j{}^1\partial x_m{}^1}\{2\gamma_{jk}(\bar{t}_1, t_0)\gamma_{ml}(\bar{t}_1, t_0) + f_m{}^1[\gamma_{jk}(\bar{t}_1, t_0)\tau_l + \gamma_{jl}(\bar{t}_1, t_0)\tau_k] + f_j{}^1[\tau_l\gamma_{mk}(\bar{t}_1, t_0) + \tau_k\gamma_{ml}(\bar{t}_1, t_0)] + 2f_j{}^1f_m{}^1\tau_k\tau_l\} + \frac{\partial^2 h_i}{\partial x_j{}^1\partial\mu_m{}^1}\Big\{[\gamma_{jk}(\bar{t}_1, t_0)\psi_{ml}(\bar{t}_1, t_0) + \gamma_{jl}(\bar{t}_1, t_0)\psi_{mk}(\bar{t}_1, t_0)] - \frac{\partial H}{\partial x_m{}^1}[\gamma_{jk}(\bar{t}_1, t_0)\tau_l + \gamma_{jl}(\bar{t}_1, t_0)\tau_k] + f_j{}^1[\tau_l\gamma_{mk}(\bar{t}_1, t_0) + \tau_k\gamma_{ml}(\bar{t}_1, t_0)] - 2f_j{}^1\frac{\partial H}{\partial x_m{}^1}\tau_k\tau_l\Big\} + \frac{\partial^2 h_i}{\partial x_j{}^1\partial\bar{t}_1}\{[\gamma_{jk}(\bar{t}_1, t_0)\tau_l + \gamma_{jl}(\bar{t}_1, t_0)\tau_k] + 2f_j{}^1\tau_k\tau_l\} + \frac{1}{2}\frac{\partial^2 h_i}{\partial\mu_j{}^1\partial\mu_m{}^1}\Big\{2\psi_{jk}(\bar{t}_1, t_0)\psi_{ml}(\bar{t}_1, t_0) - \frac{\partial H}{\partial x_m{}^1}[\psi_{jk}(\bar{t}_1, t_0)\tau_l + \psi_{jl}(\bar{t}_1, t_0)\tau_k] - \frac{\partial H}{\partial x_j{}^1}[\tau_l\psi_{mk}(\bar{t}_1, t_0) + \tau_k\psi_{ml}(\bar{t}_1, t_0)] + 2\frac{\partial H}{\partial x_j{}^1}\frac{\partial H}{\partial x_m{}^1}\tau_k\tau_l\Big\} + \frac{\partial^2 h_i}{\partial\mu_j{}^1\partial\bar{t}_1}\Big\{[\psi_{jk}(\bar{t}_1, t_0)\tau_l + \psi_{jl}(\bar{t}_1, t_0)\tau_k] - 2\frac{\partial H}{\partial x_j{}^1}\tau_k\tau_l\Big\} + \frac{\partial^2 h_i}{\partial\bar{t}_1{}^2}\tau_k\tau_l = 0 \quad (32)$$

Equations (28) are $n \times n(n+1)/2$ in number, as are Eqs (29); a total of $n^2(n+1)$ differential equations (n is the dimension of the state). (Recall that $\epsilon_{jk}{}^i = \epsilon_{kj}{}^i$ and $\theta_{jk}{}^i = \theta_{kj}{}^i$). Equations (30) are $m \times n(n+1)/2$ in number and

V. An Intercept Problem

A. Zermelo's Problem

The simple intercept problem that will be treated here is Zermelo's problem for which the linear feedback gains were previously derived by Kelley.[3] It is required to find the

§ We assume that Eqs. (30) can be uniquely solved for the quadratic gains. The conditions under which this can be done are quite complicated.

minimum time path between two points for a vehicle moving at a constant velocity relative to a medium that moves at a constant velocity relative to the target point. The situation is depicted in Fig. 1.

The vehicle path is controlled via the angle u_1. The equations of state (motion) are

$$\dot{x}_1 = V \cos u_1 + z_1 = f_1$$
$$\dot{x}_2 = V \sin u_1 + z_2 = f_2 \tag{33}$$

The Euler-Lagrange equations are

$$\dot{\mu}_1 = 0$$
$$\dot{\mu}_2 = 0 \tag{34}$$

and

$$-V\mu_1 \sin u_1 + V\mu_2 \cos u_1 = 0 \tag{35}$$

The Hamiltonian for this problem is

$$H = \mu_1(V \cos u_1 + z_1) + \mu_2(V \sin u_1 + z_2) \tag{36}$$

From Eq. (35), and with the aid of the Legendre-Clebsch condition to resolve the sign, we have

$$\sin u_1 = \mu_2(\mu_1^2 + \mu_2^2)^{-1/2}$$
$$\cos u_1 = \mu_1(\mu_1^2 + \mu_2^2)^{-1/2} \tag{37}$$

It is seen that minimum time paths for this problem are straight lines (constant u_1). The optimal nominal path is taken as the path from (0, 0) to (2, 1) where $V = 1$, $z_1 = 0$, and $z_2 = \frac{1}{2}$. The variational boundary value problem boundary conditions are

$$\bar{x}_1(\bar{t}_0) = 0 \qquad h_1 = \bar{x}_1(\bar{t}_1) - 2 = 0$$
$$\bar{x}_2(\bar{t}_0) = 0 \qquad h_2 = \bar{x}_2(\bar{t}_1) - 1 = 0 \tag{38}$$
$$(\bar{t}_0 = 0) \qquad h_3 = H[\bar{\mu}(\bar{t}_1)] - 1 = 0$$

Other pertinent data relative to the optimal nominal path are

$$\bar{\mu}_1 = 1/V \qquad \sin\bar{u}_1 = 0$$
$$\bar{\mu}_2 = 0 \qquad \cos\bar{u}_1 = 1 \tag{39}$$
$$\bar{t}_1 = 2 \qquad \bar{u}_1 = 0$$

The linear and quadratic feedback gains for Zermelo's problem are derived in Appendix B and appear in Eqs. (B11) and (B16). The linear gains are identical to those previously obtained by Kelley.[3]

The neighboring extremal feedback control law for this problem, up to second-order terms in perturbations in the state, is

$$\delta u_1(t) = \alpha(t)\delta x(t) + \tfrac{1}{2}\, \delta x^T(t)\beta^1(t)\delta x(t) \tag{40}$$

where $\delta x^T = [\delta x_1, \delta x_2]$. The linear feedback gains matrix α is given in Eq. (B11), and the matrix of quadratic feedback gains β^1 is given in Eq. (B16). Also of interest is the change in terminal (minimum) time that is given, to the second order, by

$$dt_1 = \tau\, \delta x(t_0) + \tfrac{1}{2}\, \delta x^T(t_0)\kappa\, \delta x(t_0) \tag{41}$$

where τ is given in Eq. (B12) and κ in Eq. (B17). The change in terminal time is given here in terms of deviations in the state at t_0. This may be readily generalized to any time t in $(t_0, \bar{t}_1)$.

B. Control Simulations

Prior to simulating control law (40) with continuous state error sampling, some simulations based on a single sampling time ($t_0 = 0$) were conducted. This means that a control

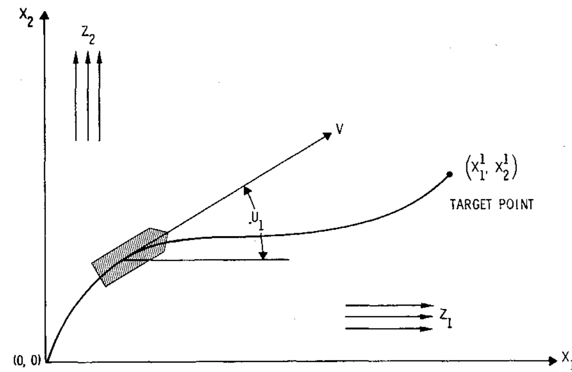

Fig. 1 Zermelo's problem.

correction is computed at $t_0 = 0$ via Eq. (40) and a straight-line path is subsequently followed (extremals for this problem are straight lines). This permits evaluation of miss distance and a comparison of linear with quadratic feedback gains effectiveness on this basis. Such a set of simulations is given in Fig. 2 for initial perturbations from the optimal nominal $\bar{x}_2(t_0)$ only. For the paths denoted by L, the control correction $\delta u_1(t_0)$ was computed using only the linear terms in Eq. (40). The paths denoted by Q employed a control correction including the quadratic feedback gains.

It is seen that the quadratic gains substantially decrease the miss distance in most cases. This is not true in all cases, in particular for large positive perturbations in $x_2(t_0)$. This result is not unexpected. The linear correction is the large one. It is difficult to assess a priori the effectiveness of the next higher-order correction that may be an "overcorrection." It will be recalled that the analysis is essentially based on Taylor series expansions. A convergent Taylor series guarantees only a finite remainder.

Control simulations with continuous error sampling are met with two difficulties inherent to the neighboring extremal control scheme. One of these is that no gains are available for times greater than the nominal terminal time. One may "run out" of a nominal trajectory. Secondly, the gains may become unbounded¶ at the nominal terminal time due to hard terminal constraints on the state, thus violating the approximations on which their derivation is based. Both of these difficulties arise in the present intercept problem and may be expected to adversely affect the terminal accuracy of the control simulations.

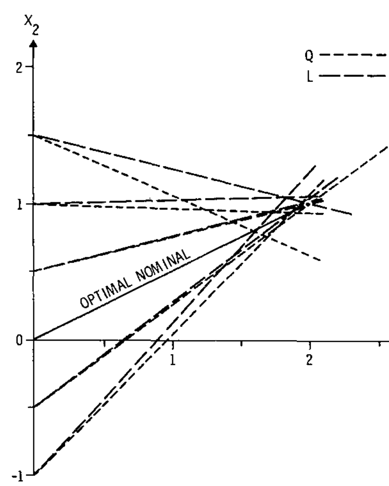

Fig. 2 Control simulations, single sampling time ($t_0 = 0$).

¶ Note that the matrix γ [Eq. (B10)] is singular at $t = t_1$, as are the linear and quadratic feedback gains [Eqs. (B11) and (B16)].

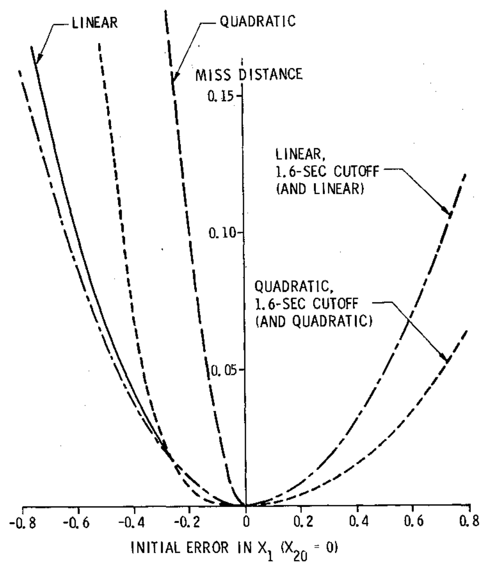

Fig. 3 Terminal accuracy for initial X_1 errors simulations.

To circumvent these difficulties, an upper bound of 60° was imposed on $|u_1|$. For times greater than two, the nominal terminal time, control is also to be held on this bound. A series of errors in the initial values of the coordinates x_1 and x_2 was simulated. The trajectories were terminated at the point of closest approach to the target (2, 1). The associated distance is called miss distance. The miss distance for simulations with both the linear and quadratic gains is given in Figs. 3 and 4 and is marked "Linear" and "Quadratic," respectively. On the basis of the criterion of miss distance, the quadratic gains are superior to linear gains only for positive initial errors in x_1. These are the only errors studied, incidentally, for which the terminal time is less than the nominal terminal time. Otherwise, the quadratic gains are inferior to linear gains for large errors.

The inconsistency of these results with the results based on a single sampling time would indicate that the terminal aspects of this control scheme are responsible. Near the ends of the trajectories the linear gains become unbounded as $1/(2 - t)$, whereas the quadratic gains become unbounded as $1/(2 - t)^2$. The quadratic gains will therefore deteriorate first and may well produce poorer terminal accuracies.

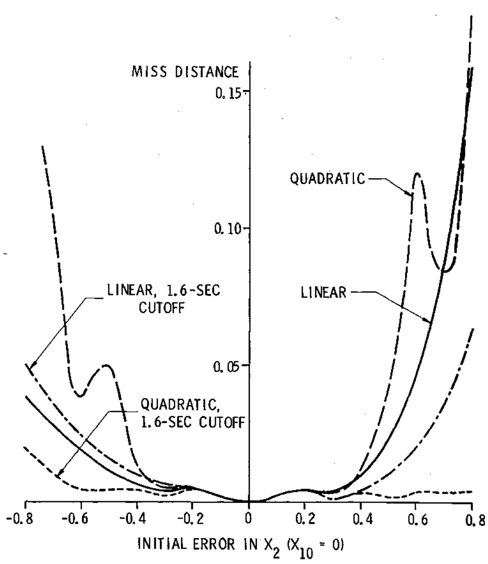

Fig. 4 Terminal accuracy for initial X_2 errors simulations.

In view of this the simulations were repeated with the following modifications. At the time of 1.6 the control, u_1 was fixed at the value $\bar{u}_1{}^{**} + \delta u_1(1.6)$ for the remainder of the trajectory. Results of these simulations for both the linear and quadratic gains are given in Figs. 3 and 4. With this modification the quadratic gains prove superior to the linear gains in all cases except large negative errors in x_1, which is the worst error situation for which both the linear and quadratic approximations are very poor. In fact, the quadratic gains substantially increase the range of initial errors that can be satisfactorily controlled.

In addition to the criterion of miss distance, the simulations should be compared on the basis of performance, which in this problem is time. In the simulations studied, the terminal time did not differ significantly between the various control schemes for small errors in the coordinates. For large errors, the terminal times were somewhat erratic, although in general quadratic gains gave smaller terminal times.

VI. Concluding Remarks

Experience with the simple control problem studied here does not justify strong conclusions as to the merits of a quadratic control approximation as opposed to a linear one. It remains to test these approximations on more complex problems. Some tentative comments based on experience to date are, however, in order.

It appears, as would be expected, that the quadratic control approximation in general more accurately represents neighboring extremals than the linear one. We expect this to be true of higher-order approximations as well. This is not true for times close to the terminal time, however. We may, in fact, expect higher-order gains to deteriorate sooner (in time) than lower-order gains. These undesirable terminal aspects of the neighboring extremal control scheme can be circumvented, as was done here, by abandoning new control correction computations before terminal time is reached. In the intercept problem treated here this, in fact, considerably enhanced the terminal accuracy of the quadratic approximation. Perhaps a sampled-data version of the present scheme, with no sampling near terminal time, would be a better approach.

From the point of view of actual control-system design, the engineer must answer the question as to whether the considerable added complexity of a high-order control scheme is justified by the improved terminal accuracy and performance. The answer to this question must be made for each control system, independently, by simulating the various control approximations.

It may be pointed out that there are two distinct undesirable terminal aspects of the present control scheme. One of these is the unavailability of gains for time greater than the nominal terminal time. This problem may perhaps be solved by making state comparisons (to form the error in state) transversely, i.e., at equal times to go for minimum time problems, as suggested by Kelley.[9,10] The other problem is that of unbounded gains at nominal terminal time, a disturbing feature to say the least. Perhaps we should not insist on "hitting a point" on the terminal surface, but settle for a small volume in the neighborhood of that point, and specify "soft" rather than "hard" constraints at the terminal point.

Appendix A

We derive here the differential equations and boundary conditions for the linear and quadratic feedback gains by the procedure outlined in Sec. IV-A. Expanding the right-hand sides of Eqs. (12) and (13) we obtain, retaining second-order terms,

** u_1 is a constant in this problem.

$$\delta\dot{x}_i = \frac{\partial f_i}{\partial x_j}\,\delta x_j + \frac{\partial f_i}{\partial u_j}\,\delta u_j + \frac{1}{2}\left[\frac{\partial^2 f_i}{\partial x_j \partial x_k}\,\delta x_j \delta x_k + 2\,\frac{\partial^2 f_i}{\partial x_j \partial u_k}\,\delta x_j \delta u_k + \frac{\partial^2 f_i}{\partial u_j \partial u_k}\,\delta u_j \delta u_k\right] \qquad i = 1, \ldots, n \tag{A1}$$

$$-\delta\dot{\mu}_i = \frac{\partial^2 H}{\partial x_i \partial x_j}\,\delta x_j + \frac{\partial^2 H}{\partial x_i \partial \mu_j}\,\delta\mu_j + \frac{\partial^2 H}{\partial x_i \partial u_j}\,\delta u_j + \frac{1}{2}\left[\frac{\partial^3 H}{\partial x_i \partial x_j \partial x_k}\,\delta x_j \delta x_k + 2\,\frac{\partial^3 H}{\partial x_i \partial x_j \partial \mu_k}\,\delta x_j \delta \mu_k + 2\,\frac{\partial^3 H}{\partial x_i \partial x_j \partial u_k}\,\delta x_j \delta u_k + 2\,\frac{\partial^3 H}{\partial x_i \partial \mu_j \partial u_k}\,\delta \mu_j \delta u_k + \frac{\partial^3 H}{\partial x_i \partial u_j \partial u_k}\,\delta u_j \delta u_k\right] \qquad i = 1, \ldots, n \tag{A2}$$

Expanding Eq. (14),

$$\frac{\partial^2 H}{\partial u_i \partial x_j}\,\delta x_j + \frac{\partial^2 H}{\partial u_i \partial \mu_j}\,\delta\mu_j + \frac{\partial^2 H}{\partial u_i \partial u_j}\,\delta u_j + \frac{1}{2}\left[\frac{\partial^3 H}{\partial u_i \partial x_j \partial x_k}\,\delta x_j \delta x_k + 2\,\frac{\partial^3 H}{\partial u_i \partial x_j \partial \mu_k}\,\delta x_j \delta \mu_k + 2\,\frac{\partial^3 H}{\partial u_i \partial x_j \partial u_k}\,\delta x_j \delta u_k + 2\,\frac{\partial^3 H}{\partial u_i \partial \mu_j \partial u_k}\,\delta \mu_j \delta u_k + \frac{\partial^3 H}{\partial u_i \partial u_j \partial u_k}\,\delta u_j \delta u_k\right] = 0 \qquad i = 1, \ldots, m \tag{A3}$$

Substituting in Eqs. (A1–A3) for δu from Eq. (6), $[\delta u = u(x, t) - u(\bar{x}, t)]$,

$$\delta\dot{x}_i = \frac{\partial f_i}{\partial x_j}\,\delta x_j + \frac{\partial f_i}{\partial u_j}\left(\alpha_{jk}\delta x_k + \frac{1}{2}\,\beta_{kl}{}^j\delta x_k \delta x_l\right) + \frac{1}{2}\left(\frac{\partial^2 f_i}{\partial x_j \partial x_k}\,\delta x_j \delta x_k + 2\,\frac{\partial^2 f_i}{\partial x_j \partial u_k}\,\delta x_j \alpha_{kl}\delta x_l + \frac{\partial^2 f_i}{\partial u_j \partial u_k}\,\alpha_{jl}\delta x_l \alpha_{km}\delta x_m\right) \tag{A4}††$$

$$-\delta\dot{\mu}_i = \frac{\partial^2 H}{\partial x_i \partial x_j}\,\delta x_j + \frac{\partial^2 H}{\partial x_i \partial \mu_j}\,\delta\mu_j + \frac{\partial^2 H}{\partial x_i \partial u_j}\left(\alpha_{jk}\delta x_k + \frac{1}{2}\,\beta_{kl}{}^j\delta x_k \delta x_l\right) + \frac{1}{2}\left(\frac{\partial^3 H}{\partial x_i \partial x_j \partial x_k}\,\delta x_j \delta x_k + 2\,\frac{\partial^3 H}{\partial x_i \partial x_j \partial \mu_k}\,\delta x_j \delta\mu_k + 2\,\frac{\partial^3 H}{\partial x_i \partial x_j \partial u_k}\,\delta x_j \alpha_{kl}\delta x_l + 2\,\frac{\partial^3 H}{\partial x_i \partial \mu_j \partial u_k}\,\partial\mu_j \alpha_{kl}\delta x_l + \frac{\partial^3 H}{\partial x_i \partial u_j \partial u_k}\,\alpha_{jl}\delta x_l \alpha_{km}\delta x_m\right) \tag{A5}$$

$$\frac{\partial^2 H}{\partial u_i \partial x_j}\,\delta x_j + \frac{\partial^2 H}{\partial u_i \partial \mu_j}\,\delta\mu_j + \frac{\partial^2 H}{\partial u_i \partial u_j}\left(\alpha_{jk}\delta x_k + \frac{1}{2}\,\beta_{kl}{}^j\delta x_k \delta x_l\right) + \frac{1}{2}\left(\frac{\partial^3 H}{\partial u_i \partial x_j \partial x_k}\,\delta x_j \delta x_k + 2\,\frac{\partial^3 H}{\partial u_i \partial x_j \partial \mu_k}\,\delta x_j \delta\mu_k + 2\,\frac{\partial^3 H}{\partial u_i \partial x_j \partial u_k}\,\delta x_j \alpha_{kl}\delta x_l + 2\,\frac{\partial^3 H}{\partial u_i \partial \mu_j \partial u_k}\,\delta\mu_j \alpha_{kl}\delta x_l + \frac{\partial^3 H}{\partial u_i \partial u_j \partial u_k}\,\alpha_{jl}\delta x_l \alpha_{km}\delta x_m\right) = 0 \tag{A6}$$

Now substituting Eqs. (19, 20, 22, and 23) into Eqs. (A4–A6),

$$\dot{\gamma}_{ij}\delta x_j{}^\circ + \frac{1}{2}\,\dot{\epsilon}_{jk}{}^i\delta x_j{}^\circ\delta x_k{}^\circ = \frac{\partial f_i}{\partial x_j}\left(\gamma_{jk}\delta x_k{}^\circ + \frac{1}{2}\,\epsilon_{kl}{}^j\delta x_k{}^\circ\delta x_l{}^\circ\right) + \frac{\partial f_i}{\partial u_j}\left[\alpha_{jk}\left(\gamma_{kl}\delta x_l{}^\circ + \frac{1}{2}\,\epsilon_{lm}{}^k\delta x_l{}^\circ\delta x_m{}^\circ\right) + \frac{1}{2}\,\beta_{kl}{}^j\gamma_{km}\delta x_m{}^\circ\gamma_{ln}\delta x_n{}^\circ\right] + \frac{1}{2}\left(\frac{\partial^2 f_i}{\partial x_j \partial x_k}\,\gamma_{jl}\delta x_l{}^\circ\gamma_{km}\,\delta x_m{}^\circ + 2\,\frac{\partial^2 f_i}{\partial x_j \partial u_k}\,\gamma_{jm}\delta x_m{}^\circ\alpha_{kl}\gamma_{ln}\delta x_n{}^\circ + \frac{\partial^2 f_i}{\partial u_j \partial u_k}\,\alpha_{jl}\gamma_{ln}\delta x_n{}^\circ\alpha_{km}\gamma_{mp}\delta x_p{}^\circ\right) \tag{A7}$$

$$-\dot{\psi}_{ij}\delta x_j{}^\circ - \frac{1}{2}\,\dot{\theta}_{jk}{}^i\delta x_j{}^\circ\delta x_k{}^\circ = \frac{\partial^2 H}{\partial x_i \partial x_j}\left(\gamma_{jk}\delta x_k{}^\circ + \frac{1}{2}\,\epsilon_{kl}{}^j\delta x_k{}^\circ\delta x_l{}^\circ\right) + \frac{\partial^2 H}{\partial x_i \partial \mu_j}\left(\psi_{jk}\delta x_k{}^\circ + \frac{1}{2}\,\theta_{kl}{}^j\delta x_k{}^\circ\delta x_l{}^\circ\right) + \frac{\partial^2 H}{\partial x_i \partial u_j}\left[\alpha_{jk}\left(\gamma_{kl}\delta x_l{}^\circ + \frac{1}{2}\,\epsilon_{lm}{}^k\delta x_l{}^\circ\delta x_m{}^\circ\right) + \frac{1}{2}\,\beta_{kl}{}^j\gamma_{km}\delta x_m{}^\circ\gamma_{ln}\delta x_n{}^\circ\right] + \frac{1}{2}\left(\frac{\partial^3 H}{\partial x_i \partial x_j \partial x_k}\,\gamma_{jl}\delta x_l{}^\circ\gamma_{km}\delta x_m{}^\circ + 2\,\frac{\partial^3 H}{\partial x_i \partial x_j \partial \mu_k}\,\gamma_{jl}\delta x_l{}^\circ\psi_{km}\delta x_m{}^\circ + 2\,\frac{\partial^3 H}{\partial x_i \partial x_j \partial u_k}\,\gamma_{jm}\delta x_m{}^\circ\alpha_{kl}\gamma_{ln}\delta x_n{}^\circ + 2\,\frac{\partial^3 H}{\partial x_i \partial \mu_j \partial u_k}\,\psi_{jm}\delta x_m{}^\circ\alpha_{kl}\gamma_{ln}\delta x_n{}^\circ + \frac{\partial^3 H}{\partial x_i \partial u_j \partial u_k}\,\alpha_{jl}\gamma_{ln}\delta x_n{}^\circ\alpha_{km}\gamma_{mp}\delta x_p{}^\circ\right) \tag{A8}$$

$$\frac{\partial^2 H}{\partial u_i \partial x_j}\left(\gamma_{jk}\delta x_k{}^\circ + \frac{1}{2}\,\epsilon_{kl}{}^j\delta x_k{}^\circ\delta x_l{}^\circ\right) + \frac{\partial^2 H}{\partial u_i \partial \mu_j}\left(\psi_{jk}\delta x_k{}^\circ + \frac{1}{2}\,\theta_{kl}{}^j\delta x_k{}^\circ\delta x_l{}^\circ\right) + \frac{\partial^2 H}{\partial u_i \partial u_j}\left[\alpha_{jk}\left(\gamma_{kl}\delta x_l{}^\circ + \frac{1}{2}\,\epsilon_{lm}{}^k\delta x_l{}^\circ\delta x_m{}^\circ\right) + \frac{1}{2}\,\beta_{kl}{}^j\gamma_{km}\delta x_m{}^\circ\gamma_{ln}\delta x_n{}^\circ\right] + \frac{1}{2}\left[\frac{\partial^3 H}{\partial u_i \partial x_j \partial x_k}\,\gamma_{jl}\delta x_l{}^\circ\gamma_{km}\delta x_m{}^\circ + 2\,\frac{\partial^3 H}{\partial u_i \partial x_j \partial \mu_k}\,\gamma_{jl}\delta x_l{}^\circ\psi_{km}\delta x_m{}^\circ + 2\,\frac{\partial^3 H}{\partial u_i \partial x_j \partial u_k}\,\gamma_{jm}\delta x_m{}^\circ\alpha_{kl}\gamma_{ln}\delta x_n{}^\circ + 2\,\frac{\partial^3 H}{\partial u_i \partial \mu_j \partial u_k}\,\psi_{jm}\delta x_m{}^\circ\alpha_{kl}\gamma_{ln}\delta x_n{}^\circ + \frac{\partial^3 H}{\partial u_i \partial u_j \partial u_k}\,\alpha_{jl}\gamma_{ln}\delta x_n{}^\circ\alpha_{km}\gamma_{mp}\delta x_p{}^\circ\right] = 0 \tag{A9}$$

We now obtain the differential equations for the linear feedback gains by equating the coefficients of $\delta x_j{}^\circ$ on both sides of Eqs. (A7) and (A8):

$$\dot{\gamma}_{ij} = \frac{\partial f_i}{\partial x_k}\,\gamma_{kj} + \frac{\partial f_i}{\partial u_k}\,\alpha_{kl}\gamma_{lj}$$

$$-\dot{\psi}_{ij} = \frac{\partial^2 H}{\partial x_i \partial x_k}\,\gamma_{kj} + \frac{\partial^2 H}{\partial x_i \partial \mu_k}\,\psi_{kj} + \frac{\partial^2 H}{\partial x_i \partial u_k}\,\alpha_{kl}\gamma_{lj} \tag{A10}$$

†† The indices m and n in this equation and in the equations to follow are often repeated in a product and are to be summed on. The range of summation for these and other indices is obvious from the dimensions of the state and control vectors, and will not be explicitly written. In addition to their use as dummy summation indices, m and n are used to denote the dimensions of the control and state vectors, respectively.

where the relationship among α_{jk}, γ_{jk}, and ψ_{jk} is obtained by equating the coefficients of δx_k° in Eq. (A9) to zero:

$$\frac{\partial^2 H}{\partial u_i \partial x_j}\gamma_{jk} + \frac{\partial^2 H}{\partial u_i \partial \mu_j}\psi_{jk} + \frac{\partial^2 H}{\partial u_i \partial u_j}\alpha_{jl}\gamma_{lk} = 0 \tag{A11}$$

Equations (A10) and (A11) can be conveniently written in matrix notation by defining

$$\alpha = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & \ddots & \vdots \\ \alpha_{m1} & \cdots & \alpha_{mn} \end{bmatrix} \quad \gamma = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1n} \\ \vdots & \ddots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{nn} \end{bmatrix} \quad \psi = \begin{bmatrix} \psi_{11} & \cdots & \psi_{1n} \\ \vdots & \ddots & \vdots \\ \psi_{n1} & \cdots & \psi_{nn} \end{bmatrix}$$

$$\left[\frac{\partial^2 H}{\partial x^2}\right] = \begin{bmatrix} \frac{\partial^2 H}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 H}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 H}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 H}{\partial x_n \partial x_n} \end{bmatrix} \quad \left[\frac{\partial^2 H}{\partial x \partial u}\right] = \begin{bmatrix} \frac{\partial^2 H}{\partial x_1 \partial u_1} & \cdots & \frac{\partial^2 H}{\partial x_1 \partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 H}{\partial x_n \partial u_1} & \cdots & \frac{\partial^2 H}{\partial x_n \partial u_m} \end{bmatrix} = \left[\frac{\partial^2 H}{\partial u \partial x}\right]^T$$

$$\left[\frac{\partial^2 H}{\partial u^2}\right] = \begin{bmatrix} \frac{\partial^2 H}{\partial u_1 \partial u_1} & \cdots & \frac{\partial^2 H}{\partial u_1 \partial u_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 H}{\partial u_m \partial u_1} & \cdots & \frac{\partial^2 H}{\partial u_m \partial u_m} \end{bmatrix} \quad \left[\frac{\partial^2 H}{\partial x \partial \mu}\right] = \begin{bmatrix} \frac{\partial^2 H}{\partial x_1 \partial \mu_1} & \cdots & \frac{\partial^2 H}{\partial x_1 \partial \mu_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 H}{\partial x_n \partial \mu_1} & \cdots & \frac{\partial^2 H}{\partial x_n \partial \mu_n} \end{bmatrix} = \left[\frac{\partial f}{\partial x}\right]^T$$

$$\left[\frac{\partial^2 H}{\partial u \partial \mu}\right] = \begin{bmatrix} \frac{\partial^2 H}{\partial u_1 \partial \mu_1} & \cdots & \frac{\partial^2 H}{\partial u_1 \partial \mu_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 H}{\partial u_m \partial \mu_1} & \cdots & \frac{\partial^2 H}{\partial u_m \partial \mu_n} \end{bmatrix} = \left[\frac{\partial f}{\partial u}\right]^T$$

We obtain

$$\alpha = -\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left(\left[\frac{\partial f}{\partial u}\right]^T \psi\gamma^{-1} + \left[\frac{\partial^2 H}{\partial u \partial x}\right]\right) \tag{A12}$$

and

$$\dot{\gamma} = \left(\left[\frac{\partial f}{\partial x}\right] - \left[\frac{\partial f}{\partial u}\right]\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left[\frac{\partial^2 H}{\partial u \partial x}\right]\right)\gamma - \left[\frac{\partial f}{\partial u}\right]\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left[\frac{\partial f}{\partial u}\right]^T\psi$$

$$\dot{\psi} = \left(\left[\frac{\partial^2 H}{\partial x \partial u}\right]\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left[\frac{\partial^2 H}{\partial u \partial x}\right] - \left[\frac{\partial^2 H}{\partial x^2}\right]\right)\gamma + \left(-\left[\frac{\partial f}{\partial x}\right]^T + \left[\frac{\partial^2 H}{\partial x \partial u}\right]\left[\frac{\partial^2 H}{\partial u^2}\right]^{-1}\left[\frac{\partial f}{\partial u}\right]^T\right)\psi \tag{A13}$$

Now we obtain the differential equations for the quadratic feedback gains by equating the collected coefficients of $\delta x_j^\circ \delta x_k^\circ$ on both sides of Eqs. (A7) and (A8), noting that $\delta x_j^\circ x_k^\circ = \delta x_k^\circ \delta x_j^\circ$:

$$\dot{\epsilon}_{jk}^i = \frac{1}{2}\dot{\epsilon}_{jk}^i + \frac{1}{2}\dot{\epsilon}_{kj}^i = \left(\frac{\partial f_i}{\partial x_l} + \frac{\partial f_i}{\partial u_m}\alpha_{ml}\right)\epsilon_{jk}^l + \frac{1}{2}\frac{\partial f_i}{\partial u_l}\beta_{mn}^l(\gamma_{mj}\gamma_{nk} + \gamma_{mk}\gamma_{nj}) + \frac{1}{2}\frac{\partial^2 f_i}{\partial x_l \partial x_m}(\gamma_{lj}\gamma_{mk} + \gamma_{lk}\gamma_{mj}) + \frac{\partial^2 f_i}{\partial x_l \partial u_m}\alpha_{mn}(\gamma_{lj}\gamma_{nk} + \gamma_{lk}\gamma_{nj}) + \frac{1}{2}\frac{\partial^2 f_i}{\partial u_l \partial u_m}\alpha_{ln}\alpha_{mp}(\gamma_{nj}\gamma_{pk} + \gamma_{nk}\gamma_{pj}) \tag{A14}$$

$$-\dot{\theta}_{jk}^i = \left(\frac{\partial^2 H}{\partial x_i \partial x_l} + \frac{\partial^2 H}{\partial x_i \partial u_m}\alpha_{ml}\right)\epsilon_{jk}^l + \frac{\partial^2 H}{\partial x_i \partial \mu_l}\theta_{jk}^l + \frac{1}{2}\frac{\partial^2 H}{\partial x_i \partial u_l}\beta_{mn}^l(\gamma_{mj}\gamma_{nk} + \gamma_{mk}\gamma_{nj}) + \frac{1}{2}\frac{\partial^3 H}{\partial x_i \partial x_l \partial x_m}(\gamma_{lj}\gamma_{mk} + \gamma_{lk}\gamma_{mj}) + \frac{\partial^3 H}{\partial x_i \partial x_l \partial \mu_m}(\gamma_{lj}\psi_{mk} + \gamma_{lk}\psi_{mj}) + \frac{\partial^3 H}{\partial x_i \partial x_l \partial u_m}\alpha_{mn}(\gamma_{lj}\gamma_{nk} + \gamma_{lk}\gamma_{nj}) + \frac{\partial^3 H}{\partial x_i \partial \mu_l \partial u_m}\alpha_{mn}(\psi_{lj}\gamma_{nk} + \psi_{lk}\gamma_{nj}) + \frac{1}{2}\frac{\partial^3 H}{\partial x_i \partial u_l \partial u_m}\alpha_{ln}\alpha_{mp}(\gamma_{nj}\gamma_{pk} + \gamma_{nk}\gamma_{pj}) \tag{A15}$$

The quadratic feedback gains β_{jk}^i are related to the other variables by equations obtained by equating the collected coefficients of $\delta x_j^\circ \delta x_k^\circ$ in Eq. (A9) to zero:

$$\left(\frac{\partial^2 H}{\partial u_i \partial x_l} + \frac{\partial^2 H}{\partial u_i \partial u_m}\alpha_{ml}\right)\epsilon_{jk}^l + \frac{\partial^2 H}{\partial u_i \partial \mu_l}\theta_{jk}^l + \frac{1}{2}\frac{\partial^2 H}{\partial u_i \partial u_l}\beta_{mn}^l(\gamma_{mj}\gamma_{nk} + \gamma_{mk}\gamma_{nj}) + \frac{1}{2}\frac{\partial^3 H}{\partial u_i \partial x_l \partial x_m}(\gamma_{lj}\gamma_{mk} + \gamma_{lk}\gamma_{mj}) + \frac{\partial^3 H}{\partial u_i \partial x_l \partial \mu_m}(\gamma_{lj}\psi_{mk} + \gamma_{lk}\psi_{mj}) + \frac{\partial^3 H}{\partial u_i \partial x_l \partial u_m}\alpha_{mn}(\gamma_{lj}\gamma_{nk} + \gamma_{lk}\gamma_{nj}) + \frac{\partial^3 H}{\partial u_i \partial \mu_l \partial u_m}\alpha_{mn}(\psi_{lj}\gamma_{nk} + \psi_{lk}\gamma_{nj}) + \frac{1}{2}\frac{\partial^3 H}{\partial u_i \partial u_l \partial u_m}\alpha_{ln}\alpha_{mp}(\gamma_{nj}\gamma_{pk} + \gamma_{nk}\gamma_{pj}) = 0 \tag{A16}$$

As indicated earlier, we obtain boundary conditions at t_0 by evaluating Eq. (19) at t_0:

$$\delta x_i(t_0) = \gamma_{ij}(t_0, t_0)\delta x_j^\circ + \tfrac{1}{2}\epsilon_{jk}{}^i(t_0, t_0)\delta x_j^\circ \delta x_k^\circ \tag{A17}$$

This implies that

$$\gamma_{ij}(t_0, t_0) = \delta_{ij} \tag{A18}$$

and

$$\epsilon_{jk}{}^i(t_0, t_0) = 0 \tag{A19}$$

where δ_{ij} is the Kronecker delta.

At the terminal point Eq. (15) must be satisfied. Expanding h in a Taylor's series we must have to second order

$$dh_i^1 = \frac{\partial h_i}{\partial x_j^1}\, dx_j^1 + \frac{\partial h_i}{\partial \mu_j^1}\, d\mu_j^1 + \frac{\partial h_i}{\partial \bar{t}_1}\, dt_1 + \frac{1}{2}\left[\frac{\partial^2 h_i}{\partial x_j^1 \partial x_k^1}\, dx_j^1 dx_k^1 + 2\,\frac{\partial^2 h_i}{\partial x_j^1 \partial \mu_k^1}\, dx_j^1 d\mu_k^1 + 2\,\frac{\partial^2 h_i}{\partial x_j^1 \partial \bar{t}_1}\, dx_j^1 dt_1 + \frac{\partial^2 h_i}{\partial \mu_j^1 \partial \mu_k^1}\, d\mu_j^1 d\mu_k^1 + 2\,\frac{\partial^2 h_i}{\partial \mu_j^1 \partial \bar{t}_1}\, d\mu_j^1 dt_1 + \frac{\partial^2 h_i}{\partial \bar{t}_1^2}\, dt_1^2\right] = 0$$

$$i = 1, \ldots, n+1 \tag{A20}$$

where

$$dx_i^1 = \delta x_i(\bar{t}_1) + \frac{d}{dt}\,(x_i + \delta x_i)_{\bar{t}_1}\, dt_1 + \frac{1}{2}\,\frac{d^2}{dt^2}\,(x_i + \delta x_i)_{\bar{t}_1}\, dt_1^2$$

$$d\mu_i^1 = \delta \mu_i(\bar{t}_1) + \frac{d}{dt}\,(\mu_i + \delta \mu_i)_{\bar{t}_1}\, dt_1 + \frac{1}{2}\,\frac{d^2}{dt^2}\,(\mu_i + \delta \mu_i)_{\bar{t}_1}\, dt_1^2 \tag{A21}$$

and more explicitly

$$dx_i^1 = \delta x_i(\bar{t}_1) + \left[f_i + \frac{\partial f_i}{\partial x_j}\,\delta x_j + \frac{\partial f_i}{\partial u_k}\,\delta u_k\right]_{\bar{t}_1} dt_1 + \frac{1}{2}\left[\frac{\partial f_i}{\partial t} + \frac{\partial f_i}{\partial x_j}\, f_j + \frac{\partial f_i}{\partial u_j}\,\dot{u}_j\right]_{\bar{t}_1} dt_1^2 \tag{A22}$$

$$d\mu_i^1 = \delta\mu_i(\bar{t}_1) - \left[\frac{\partial H}{\partial x_i} + \frac{\partial^2 H}{\partial x_i \partial x_j}\,\delta x_j + \frac{\partial^2 H}{\partial x_i \partial \mu_j}\,\delta\mu_j + \frac{\partial^2 H}{\partial x_i \partial u_j}\,\delta u_j\right]_{\bar{t}_1} dt_1 - \frac{1}{2}\left[\frac{\partial^2 H}{\partial t \partial x_i} + \frac{\partial^2 H}{\partial x_j \partial x_i}\, f_j - \frac{\partial^2 H}{\partial \mu_j \partial x_i}\,\frac{\partial H}{\partial x_j} + \frac{\partial^2 H}{\partial u_j \partial x_i}\,\dot{u}_j\right]_{\bar{t}_1} dt_1^2$$

Substituting Eqs. (A22) in Eq. (A20) we obtain

$$\begin{aligned}
dh_i^1 = {} & \frac{\partial h_i}{\partial x_j^1}\left[\delta x_j^1 + \left(f_j + \frac{\partial f_j}{\partial x_k}\,\delta x_k + \frac{\partial f_j}{\partial u_k}\,\delta u_k\right)_{\bar{t}_1} dt_1 + \frac{1}{2}\left(\frac{\partial f_j}{\partial t} + \frac{\partial f_j}{\partial x_k}\, f_k + \frac{\partial f_j}{\partial u_k}\,\dot{u}_k\right)_{\bar{t}_1} dt_1^2\right] + \\
& \frac{\partial h_i}{\partial \mu_j^1}\left[\delta\mu_j^1 - \left(\frac{\partial H}{\partial x_j} + \frac{\partial^2 H}{\partial x_j \partial x_k}\,\delta x_k + \frac{\partial^2 H}{\partial x_j \partial \mu_k}\,\delta\mu_k + \frac{\partial^2 H}{\partial x_j \partial u_k}\,\delta u_k\right)_{\bar{t}_1} dt_1 - \frac{1}{2}\left(\frac{\partial^2 H}{\partial t \partial x_j} + \frac{\partial^2 H}{\partial x_k \partial x_j}\, f_k - \frac{\partial^2 H}{\partial \mu_k \partial x_j}\,\frac{\partial H}{\partial x_k} + \frac{\partial^2 H}{\partial u_k \partial x_j}\,\dot{u}_k\right)_{\bar{t}_1} dt_1^2\right] + \\
& \frac{\partial h_i}{\partial \bar{t}_1}\, dt_1 + \frac{1}{2}\left[\frac{\partial^2 h_i}{\partial x_j^1 \partial x_k^1}\,(\delta x_j^1 + f_j^1 dt_1)(\delta x_k^1 + f_k^1 dt_1) + 2\,\frac{\partial^2 h_i}{\partial x_j^1 \partial \mu_k^1}\,(\delta x_j^1 + f_j^1 dt_1)\left(\delta\mu_k^1 - \frac{\partial H}{\partial x_k^1}\, dt_1\right) + \right. \\
& \left. 2\,\frac{\partial^2 h_i}{\partial x_j^1 \partial \bar{t}_1}\,(\delta x_j^1 + f_j^1 dt_1) dt_1 + \frac{\partial^2 h_i}{\partial \mu_j^1 \partial \mu_k^1}\left(\delta\mu_j^1 - \frac{\partial H}{\partial k_j^1}\, dt_1\right)\left(\delta\mu_k^1 - \frac{\partial H}{\partial x_k^1}\, dt_1\right) + 2\,\frac{\partial^2 h_i}{\partial \mu_j^1 \partial \bar{t}_1}\left(\delta\mu_j^1 - \frac{\partial H}{\partial x_j^1}\, dt_1\right) dt_1 + \frac{\partial^2 h_i}{\partial \bar{t}_1^2}\, dt_1^2\right] = 0
\end{aligned} \tag{A23}$$

Now eliminating δu via Eq. (6) and substituting Eqs. (19–21) in Eq. (A23),

$$\begin{aligned}
& \frac{\partial h_i}{\partial x_j^1}\left\{\gamma_{jk}(\bar{t}_1, t_0)\delta x_k^\circ + \frac{1}{2}\,\epsilon_{kl}{}^j(\bar{t}_1, t_0)\delta x_k^\circ \delta x_l^\circ + f_j^1\left(\tau_k \delta x_k^\circ + \frac{1}{2}\,\kappa_{kl}\delta x_k^\circ \delta x_l^\circ\right) + \right. \\
& \left. \left[\frac{\partial f_j}{\partial x_k^1}\,\gamma_{kl}(\bar{t}_1, t_0)\delta x_l^\circ + \frac{\partial f_j}{\partial u_k^1}\,\alpha_{kl}(\bar{x}, \bar{t}_1)\gamma_{lm}(\bar{t}_1, t_0)\delta x_m^\circ\right]\tau_n \delta x_n^\circ + \frac{1}{2}\left(\frac{\partial f_j}{\partial t} + \frac{\partial f_j}{\partial x_k}\, f_k + \frac{\partial f_j}{\partial u_k}\,\dot{u}_k\right)_{\bar{t}_1} \tau_l \delta x_l^\circ \tau_m \delta x_m^\circ\right\} + \\
& \frac{\partial h_i}{\partial \mu_j^1}\left\{\psi_{jk}(\bar{t}_1, t_0)\delta x_k^\circ + \frac{1}{2}\,\theta_{kl}{}^j(\bar{t}_1, t_0)\delta x_k^\circ \delta x_l^\circ - \frac{\partial H}{\partial x_j^1}\left(\tau_k \delta x_k^\circ + \frac{1}{2}\,\kappa_{kl}\delta x_k^\circ \delta x_l^\circ\right) - \left[\frac{\partial^2 H}{\partial x_j^1 \partial x_k^1}\,\gamma_{kl}(\bar{t}_1, t_0)\delta x_l^\circ + \right.\right. \\
& \left. \frac{\partial^2 H}{\partial x_j^1 \partial \mu_k^1}\,\psi_{kl}(\bar{t}_1, t_0)\delta x_l^\circ + \frac{\partial^2 H}{\partial x_j^1 \partial u_k^1}\,\alpha_{kl}(\bar{x}, \bar{t}_1)\gamma_{lm}(\bar{t}_1, t_0)\delta x_m^\circ\right]\tau_n \delta x_n^\circ - \frac{1}{2}\left(\frac{\partial^2 H}{\partial t \partial x_j} + \frac{\partial^2 H}{\partial x_k \partial x_j}\, f_k - \frac{\partial^2 H}{\partial \mu_k \partial x_j}\,\frac{\partial H}{\partial x_k} + \frac{\partial^2 H}{\partial u_k \partial x_j}\,\dot{u}_k\right)_{\bar{t}_1} \times \\
& \left. \tau_l \delta x_l^\circ \tau_m \delta x_m^\circ\right\} + \frac{\partial h_i}{\partial \bar{t}_1}\left(\tau_j \delta x_j^\circ + \frac{1}{2}\,\kappa_{jk}\delta x_j^\circ \delta x_k^\circ\right) + \frac{1}{2}\,\frac{\partial^2 h_i}{\partial x_j^1 \partial x_k^1}\,[\gamma_{jl}(\bar{t}_1, t_0)\delta x_l^\circ \gamma_{km}(\bar{t}_1, t_0)\delta x_m^\circ + \gamma_{jl}(\bar{t}_1, t_0)\delta x_l^\circ f_k^1 \tau_m \delta x_m^\circ + \\
& f_j^1 \tau_m\, \delta x_m^\circ \gamma_{kl}(\bar{t}_1;\, t_0)\delta x_l^\circ + f_j^1 f_k^1 \tau_m \delta x_m^\circ \tau_l \delta x_l^\circ] + \frac{\partial^2 h_i}{\partial x_j^1 \partial \mu_k^1}\left[\gamma_{jl}(\bar{t}_1, t_0)\delta x_l^\circ \psi_{km}(\bar{t}_1, t_0)\delta x_m^\circ - \gamma_{jl}(\bar{t}_1, t_0)\delta x_l^\circ\,\frac{\partial H}{\partial x_k^1}\,\tau_m \delta x_m^\circ + \right. \\
& \left. f_j^1 \tau_m \delta x_m^\circ \psi_{kl}(\bar{t}_1, t_0)\delta x_l^\circ - f_j^1 \tau_m \delta x_m^\circ\,\frac{\partial H}{\partial x_k^1}\,\tau_l \delta x_l^\circ\right] + \frac{\partial^2 h_i}{\partial x_j^1 \partial \bar{t}_1}\,[\gamma_{jk}(\bar{t}_1, t_0)\delta x_k^\circ + f_j^1 \tau_l \delta x_l^\circ]\tau_m \delta x_m^\circ + \frac{1}{2}\,\frac{\partial^2 h_i}{\partial \mu_j^1 \partial \mu_k^1} \times \\
& \left[\psi_{jl}(\bar{t}_1, t_0)\delta x_l^\circ \psi_{km}(\bar{t}_1, t_0)\delta x_m^\circ - \psi_{jl}(\bar{t}_1, t_0)\delta x_l^\circ\,\frac{\partial H}{\partial x_k^1}\,\tau_m \delta x_m^\circ - \frac{\partial H}{\partial x_j^1}\,\tau_m \delta x_m^\circ \psi_{kl}(\bar{t}_1, t_0)\delta x_l^\circ + \frac{\partial H}{\partial x_j^1}\,\frac{\partial H}{\partial x_k^1}\,\tau_l \delta x_l^\circ \tau_m \delta x_m^\circ\right] + \\
& \frac{\partial^2 h_i}{\partial \mu_j^1 \partial \bar{t}_1}\left[\psi_{jk}(\bar{t}_1, t_0)\delta x_k^\circ - \frac{\partial H}{\partial x_j^1}\,\tau_k \delta x_k^\circ\right]\tau_l \delta x_l^\circ + \frac{1}{2}\,\frac{\partial^2 h_i}{\partial \bar{t}_1^2}\,\tau_j \delta x_j^\circ \tau_k \delta x_k^\circ = 0
\end{aligned} \tag{A24}$$

Setting the collected coefficients of $\delta x_k{}^\circ$ in Eq. (A24) equal to zero, we obtain

$$\frac{\partial h_i}{\partial x_j{}^1}\,[\gamma_{jk}(\bar{t}_1, t_0) + f_j{}^1\tau_k] + \frac{\partial h_i}{\partial \mu_j{}^1}\left[\psi_{jk}(\bar{t}_1, t_0) - \frac{\partial H}{\partial x_j{}^1}\,\tau_k\right] + \frac{\partial h_i}{\partial \bar{t}_1}\,\tau_k = 0 \tag{A25}$$

Setting the collected coefficients of $\delta x_k{}^\circ \delta x_l{}^\circ$ in Eq. (A24) equal to zero yields

$$\begin{aligned}
&\frac{\partial h_i}{\partial x_j{}^1}\Bigg\{\epsilon_{kl}{}^j(\bar{t}_1, t_0) + f_j{}^1\kappa_{kl} + \frac{\partial f_j}{\partial x_n{}^1}\,[\gamma_{nk}(\bar{t}_1, t_0)\tau_l + \gamma_{nl}(\bar{t}_1, t_0)\tau_k] + \frac{\partial f_j}{\partial u_m{}^1}\,\alpha_{mn}(\bar{x}, \bar{t}_1)[\gamma_{nk}(\bar{t}_1, t_0)\tau_l + \gamma_{nl}(\bar{t}_1, t_0)\tau_k] + \\
&\quad\left(\frac{\partial f_j}{\partial t} + \frac{\partial f_j}{\partial x_m}\,f_m + \frac{\partial f_j}{\partial u_m}\,\dot{u}_m\right)_{\bar{t}_1}\tau_k\tau_l\Bigg\} + \frac{\partial h_i}{\partial \mu_j{}^1}\Bigg\{\theta_{kl}{}^j(\bar{t}_1, t_0) - \frac{\partial H}{\partial x_j{}^1}\,\kappa_{kl} - \frac{\partial^2 H}{\partial x_j{}^1\partial x_m{}^1}\,[\gamma_{mk}(\bar{t}_1, t_0)\tau_l + \gamma_{ml}(\bar{t}_1, t_0)\tau_k] - \\
&\qquad\frac{\partial^2 H}{\partial x_j{}^1\partial \mu_m{}^1}\,[\psi_{mk}(\bar{t}_1, t_0)\tau_l + \psi_{ml}(\bar{t}_1, t_0)\tau_k] - \frac{\partial^2 H}{\partial x_j{}^1\partial u_m{}^1}\,\alpha_{mn}(\bar{x}, \bar{t}_1)[\gamma_{nk}(\bar{t}_1, t_0)\tau_l + \gamma_{nl}(\bar{t}_1, t_0)\tau_k] - \\
&\quad\left(\frac{\partial^2 H}{\partial t\partial x_j} + \frac{\partial^2 H}{\partial x_m\partial x_j}\,f_m - \frac{\partial^2 H}{\partial \mu_m\partial x_j}\,\frac{\partial H}{\partial x_m} + \frac{\partial^2 H}{\partial u_m\partial x_j}\,\dot{u}_m\right)_{\bar{t}_1}\tau_k\tau_l\Bigg\} + \frac{\partial h_i}{\partial \bar{t}_1}\,\kappa_{kl} + \frac{1}{2}\,\frac{\partial^2 h_i}{\partial x_j{}^1\partial x_m{}^1}\,\{[\gamma_{jk}(\bar{t}_1, t_0)\gamma_{ml}(\bar{t}_1, t_0) + \\
&\quad\gamma_{jl}(\bar{t}_1, t_0)\gamma_{mk}(\bar{t}_1, t_0)] + f_m{}^1[\gamma_{jk}(\bar{t}_1, t_0)\tau_l + \gamma_{jl}(\bar{t}_1, t_0)\tau_k] + f_j{}^1[\tau_l\gamma_{mk}(\bar{t}_1, t_0) + \tau_k\gamma_{ml}(\bar{t}_1, t_0)] + 2f_j{}^1f_m{}^1\tau_k\tau_l\} + \\
&\qquad\frac{\partial^2 h_i}{\partial x_j{}^1\partial \mu_m{}^1}\Bigg\{[\gamma_{jk}(\bar{t}_1, t_0)\psi_{ml}(\bar{t}_1, t_0) + \gamma_{jl}(\bar{t}_1, t_0)\psi_{mk}(\bar{t}_1, t_0)] - \frac{\partial H}{\partial x_m{}^1}\,[\gamma_{jk}(\bar{t}_1, t_0)\tau_l + \gamma_{jl}(\bar{t}_1, t_0)\tau_k] + \\
&\quad f_j{}^1[\tau_l\gamma_{mk}(\bar{t}_1, t_0) + \tau_k\gamma_{ml}(\bar{t}_1, t_0)] - 2f_j{}^1\,\frac{\partial H}{\partial x_m{}^1}\,\tau_k\tau_l\Bigg\} + \frac{\partial^2 h_i}{\partial x_j{}^1\partial \bar{t}_1}\Bigg\{[\gamma_{jk}(\bar{t}_1, t_0)\tau_l + \gamma_{jl}(\bar{t}_1, t_0)\tau_k] + 2f_j{}^1\tau_k\tau_l\Bigg\} + \\
&\qquad\frac{1}{2}\,\frac{\partial^2 h_i}{\partial \mu_j{}^1\partial \mu_m{}^1}\Bigg\{[\psi_{jk}(\bar{t}_1, t_0)\psi_{ml}(\bar{t}_1, t_0) + \psi_{jl}(\bar{t}_1, t_0)\psi_{mk}(\bar{t}_1, t_0)] - \frac{\partial H}{\partial x_m{}^1}\,[\psi_{jk}(\bar{t}_1, t_0)\tau_l + \psi_{jl}(\bar{t}_1, t_0)\tau_k] - \\
&\qquad\frac{\partial H}{\partial x_j{}^1}\,[\tau_l\psi_{mk}(\bar{t}_1, t_0) + \tau_k\psi_{ml}(\bar{t}_1, t_0)] + 2\,\frac{\partial H}{\partial x_j{}^1}\,\frac{\partial H}{\partial x_m{}^1}\,\tau_k\tau_l\Bigg\} + \frac{\partial^2 h_i}{\partial \mu_j{}^1\partial \bar{t}_1}\Bigg\{[\psi_{jk}(\bar{t}_1, t_0)\tau_l + \psi_{jl}(\bar{t}_1, t_0)\tau_k] - \\
&\qquad\qquad 2\,\frac{\partial H}{\partial x_j{}^1}\,\tau_k\tau_l\Bigg\} + \frac{\partial^2 h_i}{\partial \bar{t}_1{}^2}\,\tau_k\tau_l = 0
\end{aligned} \tag{A26}$$

Appendix B

Here we obtain the linear and quadratic feedback gains for Zermelo's problem outlined in Sec. V. The differential equations (24) associated with the linear feedback gains are

$$\dot{\gamma} = \begin{bmatrix} 0 & 0 \\ 0 & V^2 \end{bmatrix}\psi \qquad \dot{\psi} = 0 \tag{B1}$$

and the linear feedback gains [Eq. (27)] are

$$\alpha = [0 \qquad V]\psi\gamma^{-1} \tag{B2}$$

The differential equations [(28) and (29)] associated with the quadratic feedback gains reduce to

$$\dot{\epsilon}_{jk}{}^i = \frac{\partial f_i}{\partial u_1}\,\alpha_{1l}\epsilon_{jk}{}^l + \frac{\partial f_i}{\partial u_1}\,\beta_{mn}{}^1\gamma_{mj}\gamma_{nk} + \frac{\partial^2 f_i}{\partial u_1{}^2}\,\alpha_{1n}\alpha_{1p}\gamma_{nj}\gamma_{pk} \qquad \dot{\theta}_{jk}{}^i = 0 \tag{B3}$$

and the quadratic feedback gains [Eq. (30)] are given by

$$-\alpha_{1l}\epsilon_{jk}{}^l + V\theta_{jk}{}^2 - \beta_{mn}{}^1\gamma_{mj}\gamma_{nk} - V\alpha_{1n}(\psi_{1j}\gamma_{nk} + \psi_{1k}\gamma_{nj}) = 0 \tag{B4}$$

Equations (B3) may be written in the expanded form

$$\begin{aligned}
\dot{\epsilon}_{jk}{}^1 &= -V\alpha_{1n}\alpha_{1p}\gamma_{nj}\gamma_{pk} \\
\dot{\epsilon}_{jk}{}^2 &= V\alpha_{1l}\epsilon_{jk}{}^l + V\beta_{mn}{}^1\gamma_{mj}\gamma_{nk} \\
\dot{\theta}_{jk}{}^i &= 0
\end{aligned} \tag{B5}$$

The quadratic feedback gains are eliminated from Eq. (B5) via Eq. (B4)

$$\begin{aligned}
\dot{\epsilon}_{jk}{}^1 &= -V\alpha_{1n}\alpha_{1p}\gamma_{nj}\gamma_{pk} \\
\dot{\epsilon}_{jk}{}^2 &= V^2\theta_{jk}{}^2 - V^2\alpha_{1n}(\psi_{1j}\gamma_{nk} + \psi_{1k}\gamma_{nj}) \\
\dot{\theta}_{jk}{}^i &= 0
\end{aligned} \tag{B6}$$

The terminal boundary conditions, in the form of Eq. (A23), are

$$\begin{aligned}
dh_1{}^1 &= \delta x_1{}^1 + V dt_1 = 0 \\
dh_2{}^1 &= \delta x_2{}^1 + z_2 dt_1 + V\delta u_1{}^1 dt_1 = 0 \\
dh_3{}^1 &= V\delta\mu_1{}^1 + z_2\delta\mu_2{}^1 + \tfrac{1}{2}V(\delta\mu_2{}^1)^2 = 0
\end{aligned} \tag{B7}$$

Equations (26) yield

$$\begin{aligned}
V\gamma_{2k}(\bar{t}_1, t_0) - z_2\gamma_{1k}(\bar{t}_1, \bar{t}_0) &= 0 \\
V\psi_{1k}(\bar{t}_1, t_0) + z_2\psi_{2k}(\bar{t}_1, \bar{t}_0) &= 0 \\
\tau_k &= -\frac{1}{V}\gamma_{1k}(\bar{t}_1, t_0)
\end{aligned} \tag{B8}$$

Equations (32) yield

$$\begin{aligned}
&\epsilon_{kl}{}^2(\bar{t}_1, t_0) - \frac{z_2}{V}\,\epsilon_{kl}{}^1(\bar{t}_1, t_0) - \\
&\qquad \alpha_{1n}(\bar{x}, \bar{t}_1)[\gamma_{nk}(\bar{t}_1, t_0)\gamma_{1l}(\bar{t}_1, t_0) + \gamma_{nl}(\bar{t}_1, t_0)\gamma_{1k}(\bar{t}_1, t_0)] = 0 \\
&V\theta_{kl}{}^1(\bar{t}_1, t_0) + z_2\theta_{kl}{}^2(\bar{t}_1, t_0) + V\psi_{2k}(\bar{t}_1, t_0)\psi_{2l}(\bar{t}_1, t_0) = 0 \\
&\kappa_{kl} = -\frac{1}{V}\,\epsilon_{kl}{}^1(\bar{t}_1, t_0)
\end{aligned} \tag{B9}$$

Solving Eqs. (B1) subject to boundary conditions (25) and (B8), we obtain

$$\gamma = \begin{bmatrix} 1 & 0 \\ \dfrac{z_2 t}{\bar{t}_1 V} & \dfrac{\bar{t}_1 - t}{\bar{t}_1} \end{bmatrix} \qquad \psi = \begin{bmatrix} -\dfrac{z_2{}^2}{\bar{t}_1 V^4} & \dfrac{z_2}{\bar{t}_1 V^3} \\ \dfrac{z_2}{\bar{t}_1 V^3} & -\dfrac{1}{\bar{t}_1 V^2} \end{bmatrix} \tag{B10}$$

The linear feedback gains [Eq. (B2)] are

$$\alpha = \left[\frac{z_2}{V^2(\bar{t}_1 - t)} \qquad -\frac{1}{V(\bar{t}_1 - t)}\right] \tag{B11}$$

and

$$\tau = \begin{bmatrix} -\dfrac{1}{V} \\ 0 \end{bmatrix} \tag{B12}$$

Having obtained the linear feedback gains solution, Eqs. (B6) may be written more explicitly

$$\dot{\epsilon}^1 = \begin{bmatrix} \dot{\epsilon}_{11}{}^1 & \dot{\epsilon}_{12}{}^1 \\ \dot{\epsilon}_{21}{}^1 & \dot{\epsilon}_{22}{}^1 \end{bmatrix} = \frac{1}{\bar{t}_1{}^2 V} \begin{bmatrix} -\dfrac{z_2{}^2}{V^2} & \dfrac{z_2}{V} \\ \dfrac{z_2}{V} & -1 \end{bmatrix}$$

$$\dot{\epsilon}^2 = V^2\theta^2 + \frac{2z_2}{\bar{t}_1{}^2 V^2} \begin{bmatrix} \dfrac{z_2{}^2}{V^2} & -\dfrac{z_2}{V} \\ -\dfrac{z_2}{V} & 1 \end{bmatrix} \tag{B13}$$

$$\dot{\theta}^1 = \begin{bmatrix} \dot{\theta}_{11}{}^1 & \dot{\theta}_{12}{}^1 \\ \dot{\theta}_{21}{}^1 & \dot{\theta}_{22}{}^1 \end{bmatrix} = 0 \qquad \dot{\theta}^2 = 0$$

The solutions of Eqs. (B13), subject to boundary conditions (31) and (B9), are

$$\left.\begin{aligned} \epsilon^1 &= \frac{t}{\bar{t}_1{}^2 V} \begin{bmatrix} -\dfrac{z_2{}^2}{V^2} & \dfrac{z_2}{V} \\ \dfrac{z_2}{V} & -1 \end{bmatrix} \\ \epsilon^2 &= \frac{t}{\bar{t}_1{}^2 V} \begin{bmatrix} \dfrac{z_2}{V}\left(2 - \dfrac{z_2{}^2}{V^2}\right) & \left(\dfrac{z_2{}^2}{V^2} - 1\right) \\ \left(\dfrac{z_2{}^2}{V^2} - 1\right) & -\dfrac{z_2}{V} \end{bmatrix} \\ \theta^1 &= \frac{1}{\bar{t}_1{}^2 V^4} \begin{bmatrix} \dfrac{z_2{}^2}{V}\left(\dfrac{3z_2{}^2}{V^2} - \dfrac{1}{V} - 2\right) & z_2\left(1 + \dfrac{1}{V} - \dfrac{3z_2{}^2}{V^2}\right) \\ z_2\left(1 + \dfrac{1}{V} - \dfrac{3z_2{}^2}{V^2}\right) & \dfrac{3z_2{}^2}{V} - 1 \end{bmatrix} \\ \theta^2 &= \frac{1}{\bar{t}_1{}^2 V^3} \begin{bmatrix} \dfrac{z_2}{V}\left(2 - \dfrac{3z_2{}^2}{V^2}\right) & \dfrac{3z_2{}^2}{V^2} - 1 \\ \dfrac{3z_2{}^2}{V^2} - 1 & -\dfrac{3z_2}{V} \end{bmatrix} \end{aligned}\right\} \tag{B14}$$

Equations (B4) reduce to

$$\begin{aligned} \beta_{11}{}^1 + \frac{2z_2 t}{\bar{t}_1 V}\beta_{12}{}^1 + \frac{z_2{}^2 t^2}{\bar{t}_1{}^2 V^2}\beta_{22}{}^1 &= -\frac{z_2{}^3}{\bar{t}_1{}^2 V^5} + \frac{2z_2}{\bar{t}_1 V^3(\bar{t}_1 - t)} \\ \beta_{12}{}^1 + \frac{z_2 t}{\bar{t}_1 V}\beta_{22}{}^1 &= \frac{z_2{}^2}{\bar{t}_1 V^4(\bar{t}_1 - t)} - \frac{1}{V^2(\bar{t}_1 - t)^2} \\ \beta_{22}{}^1 &= -\frac{z_2}{V^3(\bar{t}_1 - t)^2} \end{aligned} \tag{B15}$$

and the quadratic feedback gains are

$$\beta^1 = \begin{bmatrix} \beta_{11}{}^1 & \beta_{12}{}^1 \\ \beta_{21}{}^1 & \beta_{22}{}^1 \end{bmatrix} = \frac{1}{V^3(\bar{t}_1 - t)^2} \begin{bmatrix} \dfrac{2z_2V^2 - z_2{}^3}{V^2} & \dfrac{z_2{}^2 - V^2}{V} \\ \dfrac{z_2{}^2 - V^2}{V} & -z_2 \end{bmatrix} \tag{B16}$$

The coefficients κ_{ij} [Eq. (B9)] are

$$\kappa = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix} = \frac{1}{\bar{t}_1 V^2} \begin{bmatrix} \dfrac{z_2{}^2}{V^2} & -\dfrac{z_2}{V} \\ -\dfrac{z_2}{V} & 1 \end{bmatrix} \tag{B17}$$

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